MATH 2050 - Subsequential limits, limsup & liminf

(Reference: Bartle § 3.4)

Q: Given a bdd seq. (x_n), what is

$$\mathcal{L} := \{ \ell \in \mathbb{R} \mid \exists \text{ subseq. } (x_{n_k}) \text{ of } (x_n) \text{ s.t. } \lim_{k\to\infty} (x_{n_k}) = \ell \} \ ?$$

Example: If $\lim (x_n) = x$, then $\mathcal{L} = \{x\}$.

Example: $(x_n) = ((-1)^n)$, then $\mathcal{L} = \{1, -1\}$

Remark: BWT $\Rightarrow \mathcal{L} \neq \phi$.

(x_n) bdd $\Rightarrow \exists M > 0$ s.t. $|x_n| \leq M \quad \forall n \in \mathbb{N}$.

So, any convergent subseq. (x_{n_k}) satisfy

$$-M \leq x_{n_k} \leq M \qquad \forall k \in \mathbb{N}.$$

$$\overset{\text{as } k\to\infty}{\Longrightarrow} \qquad -M \leq \ell \leq M$$

ie $\overset{\phi \neq}{\ }\overset{\text{BWT}}{\mathcal{L}} \subseteq [-M, M]$ is bdd subset of $\mathbb{R}$.

By completeness of $\mathbb{R}$, $\inf \mathcal{L}$, $\sup \mathcal{L}$ both exist.

Def$^{\underline{n}}$:

$$\limsup (x_n) = \overline{\lim} (x_n) := \sup \mathcal{L}$$

$$\liminf (x_n) = \underline{\lim} (x_n) := \inf \mathcal{L}$$

Thm: Let (x_n) be a bdd seq. Define another seq. (u_m) by

$$u_m := \sup\{x_n \mid n \geq m\} \quad \text{for each } m = 1, 2, 3, \ldots$$

THEN. (u_m) is a decreasing seq. with

$$\lim_{m\to\infty}(u_m) = \inf\{u_m \mid m \in \mathbb{N}\} = \overline{\lim}(x_n)$$

Proof: [Recall: $S_1 \subseteq S_2 \Rightarrow \sup S_1 \leq \sup S_2$]

$$(x_n) = (x_1, x_2, x_3, x_4, x_5, \ldots, x_n, \ldots)$$

sup $= u_1$ sup $= u_2$

$\forall m \in \mathbb{N}$. $\{x_n \mid n \geq m\} \supseteq \{x_n \mid n \geq m+1\}$

take sup. $u_m \geq u_{m+1}$

So, (u_m) is decreasing, and bdd ($\because (x_n)$ bdd)

By MCT, $\lim\limits_{m\to\infty}(u_m) = \inf\{u_m : m \in \mathbb{N}\}$.

Claim 1: $\overline{\lim}(x_n) \leq \lim(u_m)$

Pf: Recall $\overline{\lim}(x_n) = \sup \mathcal{L}$. Let $\ell \in \mathcal{L}$, then by def$^{\underline{n}}$.

$\exists$ subseq. $(x_{n_k}) \to \ell$. By def$^{\underline{n}}$ of u_m (when $m = n_k$)

$$x_{n_k} \leq u_{n_k} := \sup\{x_n \mid n \geq n_k\} \quad \forall k \in \mathbb{N}$$

let $k \to \infty$. $\ell \leq \lim\limits_{k\to\infty}(u_{n_k}) = \lim\limits_{m\to\infty}(u_m)$

(u_{n_k}) is a subseq. of the convergent seq. (u_m).

Claim 2: $\overline{\lim}(x_n) \geq \lim(u_m)$

Pf: Want to show: $\lim(u_m) \in \mathcal{L}$

We have to find a subseq (x_{n_k}) of (x_n) st

$$(x_{n_k}) \to \lim(u_m)$$

- Choose $n_1 \geq 1$ s.t. $u_1 - 1 < x_{n_1} \leq u_1 := \sup\{x_n \mid n \geq 1\}$
- Choose $n_2 > n_1$ s.t. $u_{n_1+1} - \frac{1}{2} < x_{n_2} \leq u_{n_1+1} := \sup\{x_n \mid n \geq n_1+1\}$

Do it inductively, we can choose $n_1 < n_2 < n_3 < \cdots$

st $$u_{n_k+1} - \frac{1}{k+1} < x_{n_{k+1}} \leq u_{n_k+1} \quad \forall k \in \mathbb{N}$$

Take $k \to \infty$ above, by Squeeze Thm.

$$\ell = \lim(x_{n_k}) = \lim(u_m) \in \mathcal{L}$$

$\blacksquare$

Remarks

(i) $\overline{\lim}(x_n)$, $\underline{\lim}(x_n) \in \mathcal{L}$

(ii) $\overline{\lim}(x_n)$, $\underline{\lim}(x_n)$ always exist [BUT not $\lim(x_n)$]

provided that (x_n) is bdd.

(iii) $\overline{\lim}(x_n + y_n) \leq \overline{\lim}(x_n) + \overline{\lim}(y_n)$ Pf: Exercise!

(c.f. Limit thms)